THE IMBEDDING PROBLEM FOR SYSTEMS WITH AN INCOMPLETE, COMMUTATIVE ADDITION

BY

KE-CHAN KUO

B.S., Chiao Tung University, 1940
M.S., University of Illinois, 1947

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE COLLEGE OF THE UNIVERSITY OF ILLINOIS, 1950

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THE IMBEDDING PROBLEM FOR SYSTEMS WITH AN INCOMPLETE, COMMUTATIVE ADDITION

We shall use A to designate a commutative add which is a non-vacuous set of elements equipped with an incomplete, commutative addition. The purpose of this investigation is to find the criteria for the imbeddability of commutative adds into groups.

A is an amalgam of its subgroups if (1) A is the join of its subgroups, (2) the sum of two elements exists only if they are contained in a subgroup of A, and (3) the intersection of any two subgroups of A is their common subgroup.

An ordered n-tuple $(v_1,\ldots,v_n)$, with v_i in A, is a vector of length n over A. We define the addition of vectors by the rule: $(v_1,\ldots,v_n)+(u_1,\ldots,u_m)=(v_1,\ldots,v_n,u_1,\ldots,u_m)$. The vector $(v_1,\ldots,v_{i-1},v_i+v_{i+1},v_{i+2},\ldots,v_n)$ is called a direct reduction of $(v_1,\ldots,v_n)$. A vector u is a reduction of another vector v if u is obtained by a sequence of direct reductions from v. If $v(0),\ldots,v(k)$ is a finite set of vectors with the property that v(i-1) is either a reduction of, or reducible to, v(i) for $i=1,\ldots,k$, we say that v(0) is similar to v(k).

If $u_1,\ldots,u_n$ is a permutation of $v_1,\ldots,v_n$, for v_i in A, then the vector $(u_1,\ldots,u_n)$ is said to be equivalent to the vector $(v_1,\ldots,v_n)$. If u is equivalent v, and if w is a direct re-

duction of u; then w is a direct c-reduction of v. A vector, obtained by a sequence of direct c-reductions from another, is called a c-reduction of the later. If v(i-1) is either c-reducible to, or a c-reduction of, v(i) for i = 1,...,k; we say v(0) and v(k) are c-similar.

Let $\langle v \rangle$ be the class of all vectors similar to v. Let D(A) be the set of all such classes of similar vectors over A. Define the addition in D(A) by the rule: $\langle v \rangle + \langle u \rangle = \langle v+u \rangle$. Then D(A) is an additive manifold. In particular, D(A) is a group if A is an amalgam. Similarly, let $[v]$ be the class of vectors c-similar to v. Form the set Dc(A) of all such c-similar classes. Then Dc(A) is a commutative additive manifold. Dc(A) is an abelian group if A is an amalgam.

The natural mapping $\langle v \rangle \longrightarrow [v]$ is a homomorphism of D(A) upon Dc(A). If D(A) is a group, then the kernel of this natural homomorphism is the commutator subgroup of D(A).

A is imbeddable into a group if the natural mapping $a \longrightarrow \langle (a) \rangle$ is an isomorphism of A into D(A). A is imbeddable into an abelian group if the natural mapping $a \longrightarrow [(a)]$ is an isomorphism of A into Dc(A).

The usual associative law can be expressed in various ways in

the Theory of Adds as follows.

(N.1) If (a) is similar to (b), then $a = b$.

(N.2) If (a,b) is similar to (c), then $a + b = c$.

(N.3) If v is similar to (a), then v is reducible to (a).

(N.4) If the irreducible vectors u and v are similar, then their lengths are equal. If the length is one, then $u = v$.

(N.5) If the irreducible vectors u and v are similar, then they are equal.

Similarly, define (C.i), for $i = 1,\ldots,5$, by replacing the terms "similar", "reducible", and "irreducible" in (N.i) respectively by "c-similar", "c-reducible", and "c-irreducible"; and replacing the last word "equal" in (N.5) by "equivalent". Clearly, (X.i-1) is a consequence of (X.i) for $i = 2,\ldots,5$, where X stands for N or C. The imbeddability is equivalent to (X.2).

The properties (N.3) to (N.5) had been investigated by R. Baer*. The main results found in this investigation are given as follows.

(1) (N.i) is a consequence of (C.i) for $i \leq 2$, while (C.i) is a consequence of (N.i) for $i = 3,4$, and 5.

(2) An amalgam A satisfies (C.3) if, and only if, it satisfies the following three properties.

I If $x + y$, $y + z$, and $(x + y) + z$ exist, then $x + (y + z)$ exists.

II If both $(x + y) + z$ and $x + (y + z)$ exist, they are equal.

(Q) If $(x_1-x_2,\ldots,x_{n-1}-x_n,x_n-x_1)$ exists, it is c-reducible to (0).

(3) An amalgam A satisfies (N.3) if, and only if, it satisfies (C.3) and the following two properties.

IV If $x+y$, $y+z$, and $z+x$ exist, then $(x+y)+z$ exists.

X If $x+y$, $y+z$, $z+w$, and $w+x$ exist, then at least one of $x+z$ and $y+w$ must exist.

(4) An amalgam A satisfies (C.4) if, and only if, it satisfies I, II and the following two properties.

V If $x+y$, $y+z$, and $z+w$ exist, and $x+(y+z)$ does not exist; then at least one of $(y+z)+w$ and $x+w$ must exist.

VI If $x+y$, $y+z$, $z+w$, and $w+x$ exist, then at least one of $x+z$, $(w+x)+y$, and $(y+z)+w$ must exist.

(5) (N.4) is equivalent to (C.4) and IV in amalgams.

(6) (C.5) is equivalent to II and the following property.

VII If $x+y$ and $y+z$ exist, and $(x+y)+z$ does not exist; then either $x+y=x$ and $y+z=z$, or $x=z$.

(7) (N.5) and (C.5) are equivalent in amalgams.

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* See R. Baer:"Free sums of groups and their generalizations. An analysis of the associative law," American Journal of Mathematics, Vol. 71 (1949), pp. 706-742; and two subsequent papers to be published.

VITA

Ke-chan Kuo was born on January 30, 1917, in Anking, Anhwei, China. He received the degree of Bachelor of Science in Civil Engineering from the Tangshan Engineering College of Chiao Tung University, China, in September, 1940. After winning a competitive examination given by the Chinese Government, he received a scholarship for study abroad. He entered the Graduate College of the University of Illinois in 1946 and received the degree of Master of Science in June, 1947. He transferred his major field to Mathematics in 1948 and held a part time assistantship in the Department of Mathematics from September, 1948, to January, 1949. Since then, he has been a Fellow in Mathematics of the same University. He is a member of the American Mathematical Society, Pi Mu Epsilon, Phi Kappa Phi, and Sigma Xi.

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